

# CP Phases of Neutrino Mixing in a Supersymmetric $B - L$ Gauge Model with $T_7$ Lepton Flavor Symmetry

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## Abstract

In a recently proposed renormalizable model of neutrino mixing using the non-Abelian discrete symmetry  $T_7$  in the context of a supersymmetric extension of the Standard Model with gauged  $U(1)_{B-L}$ , a correlation was obtained between  $\theta_{13}$  and  $\theta_{23}$  in the case where all four parameters are real. Here we consider one parameter to be complex, thus allowing for one Dirac CP phase  $\delta_{CP}$  and two Majorana CP phases  $\alpha_{1,2}$ . We find a slight modification to this correlation as a function of  $\delta_{CP}$ . For a given set of input values of  $\Delta m_{21}^2$ ,  $\Delta m_{32}^2$ ,  $\theta_{12}$ , and  $\theta_{13}$ , we obtain  $\sin^2 2\theta_{23}$  and  $m_{ee}$  (the effective Majorana neutrino mass in neutrinoless double beta decay) as functions of  $\tan \delta_{CP}$ . We find that the structure of this model always yields small  $|\tan \delta_{CP}|$ .

The most general  $3 \times 3$  Majorana neutrino mass matrix has six complex entries, i.e. twelve parameters. Three are overall phases of the mass eigenstates which are unobservable. The nine others are three masses, three mixing angles, and three phases: one Dirac phase  $\delta_{CP}$ , i.e. the analog of the one complex phase of the  $3 \times 3$  quark mixing matrix, and two relative Majorana phases  $\alpha_{1,2}$  for two of the three mass eigenstates. The existence of nonzero  $\delta_{CP}$  or  $\alpha_{1,2}$  means that  $CP$  conservation is violated. It is one of most important issues of neutrino physics yet to be explored experimentally.

The application of the non-Abelian discrete symmetry  $A_4$  [1] (and others) to neutrino mixing has been successful in explaining tribimaximal mixing, i.e.  $\sin^2 \theta_{12} = 1/3$ ,  $\sin^2 \theta_{23} = 1/2$ , and  $\theta_{13} = 0$ . In particular, a generic three-parameter  $A_4$  model [2] predicts all of the above with  $\delta_{CP} = \alpha_{1,2} = 0$ , leaving the three neutrino masses arbitrary. Recently, the first evidence that  $\theta_{13} \neq 0$  has been published [3] by the T2K Collaboration, i.e.

$$0.03(0.04) \leq \sin^2 2\theta_{13} \leq 0.28(0.34) \quad (1)$$

for  $\delta_{CP} = 0$  and normal (inverted) hierarchy of neutrino masses. Slightly different but similar ranges are obtained for nonzero values of  $\delta_{CP}$ . More recently, the Double Chooz Collaboration has also reported [4] a measurement of

$$\sin^2 2\theta_{13} = 0.086 \pm 0.041(\text{stat}) \pm 0.030(\text{syst}). \quad (2)$$

Their best fit is obtained by minimizing its  $\chi^2$  as a function of  $\delta_{CP}$ . However, all  $\delta_{CP}$  values are allowed within one standard deviation. One month ago, the first precise measurement of  $\sin^2 2\theta_{13}$  was announced by the Daya Bay Collaboration [5]:

$$\sin^2 2\theta_{13} = 0.092 \pm 0.016(\text{stat}) \pm 0.005(\text{syst}), \quad (3)$$

based only on a rate analysis, resulting in a  $5.2\sigma$  effect. It has been followed by the announcement this month of the RENO Collaboration [6]:

$$\sin^2 2\theta_{13} = 0.103 \pm 0.013(\text{stat}) \pm 0.011(\text{syst}), \quad (4)$$

again based only on a rate analysis, resulting in a  $6.3\sigma$  effect.

To account for  $\theta_{13} \neq 0$ , the original  $A_4$  proposal has to be modified [7]. Similarly, the original supersymmetric  $B - L$  gauge model with  $T_7$  lepton flavor symmetry [8] (which obtained tribimaximal mixing) has to be replaced as well [9]. In that latter paper, it is shown that a neutrino mass matrix with four parameters allow a nonzero  $\theta_{13}$ . Assuming that all four parameters are real, thus requiring two conditions among the six observables, i.e. the three masses and three mixing angles, the prediction

$$\sin^2 2\theta_{23} \simeq 1 - \frac{1}{2} \sin^2 2\theta_{13} \quad (5)$$

is obtained. This scenario applies of course only to the case  $\delta_{CP} = \alpha_{1,2} = 0$ . Here we consider instead the case where one parameter is complex. We then have five real parameters to describe the three masses, the three mixing angles, and the three phases. Given five inputs, we should then be able to predict the other four parameters.

The tetrahedral group  $A_4$  (12 elements) is the smallest group with a real  $\underline{3}$  representation. The Frobenius group  $T_7$  (21 elements) is the smallest group with a pair of complex  $\underline{3}$  and  $\underline{3}^*$  representations. It is generated by

$$a = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^4 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (6)$$

where  $\rho = \exp(2\pi i/7)$ , so that  $a^7 = 1$ ,  $b^3 = 1$ , and  $ab = ba^4$ . The character table of  $T_7$  (with  $\xi = -1/2 + i\sqrt{7}/2$ ) is given below.

class	$n$	$h$	$\chi_1$	$\chi_{1'}$	$\chi_{1''}$	$\chi_3$	$\chi_{3^*}$
$C_1$	1	1	1	1	1	3	3
$C_2$	7	3	1	$\omega$	$\omega^2$	0	0
$C_3$	7	3	1	$\omega^2$	$\omega$	0	0
$C_4$	3	7	1	1	1	$\xi$	$\xi^*$
$C_5$	3	7	1	1	1	$\xi^*$	$\xi$

Table 1: Character table of  $T_7$ .

The group multiplication rules of  $T_7$  include

$$\underline{3} \times \underline{3} = \underline{3}^*(23, 31, 12) + \underline{3}^*(32, 13, 21) + \underline{3}(33, 11, 22), \quad (7)$$

$$\begin{aligned} \underline{3} \times \underline{3}^* &= \underline{3}(21^*, 32^*, 13^*) + \underline{3}^*(12^*, 23^*, 31^*) + \underline{1}(11^* + 22^* + 33^*) \\ &+ \underline{1}'(11^* + \omega 22^* + \omega^2 33^*) + \underline{1}''(11^* + \omega^2 22^* + \omega 33^*). \end{aligned} \quad (8)$$

Note that  $\underline{3} \times \underline{3} \times \underline{3}$  has two invariants and  $\underline{3} \times \underline{3} \times \underline{3}^*$  has one invariant.

We now follow Ref. [9] in deriving the neutrino mass matrix. Under  $T_7$ , let  $L_i = (\nu, l)_i \sim \underline{3}$ ,  $l_i^c \sim \underline{1}, \underline{1}', \underline{1}''$ ,  $i = 1, 2, 3$ ,  $\Phi_i = (\phi^+, \phi^0)_i \sim \underline{3}$ , and  $\Phi'_i = (\phi'^0, -\phi'^-)_i \sim \underline{3}^*$ . The Yukawa couplings  $L_i l_j^c \Phi'_k$  generate the charged-lepton mass matrix

$$M_l = \begin{pmatrix} f_1 v'_1 & f_2 v'_1 & f_3 v'_1 \\ f_1 v'_2 & \omega^2 f_2 v'_2 & \omega f_3 v'_2 \\ f_1 v'_3 & \omega f_2 v'_3 & \omega^2 f_3 v'_3 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \begin{pmatrix} f_1 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & f_3 \end{pmatrix} v, \quad (9)$$

if  $v'_1 = v'_2 = v'_3 = v'/\sqrt{3}$ , as in the original  $A_4$  proposal [1].

Let  $\nu_i^c \sim \underline{3}^*$ , then the Yukawa couplings  $L_i \nu_j^c \Phi_k$  are allowed, with

$$M_D = f_D \begin{pmatrix} 0 & v_1 & 0 \\ 0 & 0 & v_2 \\ v_3 & 0 & 0 \end{pmatrix} = \frac{f_D v}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (10)$$

for  $v_1 = v_2 = v_3 = v/\sqrt{3}$  which is necessary for consistency since  $v'_1 = v'_2 = v'_3 = v'/\sqrt{3}$  has already been assumed for  $M_l$ . Note that  $\Phi$  and  $\Phi'$  have  $B - L = 0$ , and both are necessary because of supersymmetry. However, the analysis of neutrino mixing does not involve these extra supersymmetric partners.

Now add the neutral electroweak Higgs singlets  $\chi_i \sim \underline{3}$  and  $\eta_i \sim \underline{3}^*$ , both with  $B - L = -2$ . Then there are two Yukawa invariants:  $\nu_i^c \nu_j^c \chi_k$  and  $\nu_i^c \nu_j^c \eta_k$  (which has to be symmetric in  $i, j$ ). Note that  $\chi_i^* \sim \underline{3}^*$  is not the same as  $\eta_i \sim \underline{3}^*$  because they have different  $B - L$ . This means that both  $B - L$  and the complexity of the  $\underline{3}$  and  $\underline{3}^*$  representations in  $T_7$  are required for this scenario. The heavy Majorana mass matrix for  $\nu^c$  is then

$$M_{\nu^c} = h \begin{pmatrix} u_2 & 0 & 0 \\ 0 & u_3 & 0 \\ 0 & 0 & u_1 \end{pmatrix} + h' \begin{pmatrix} 0 & u'_3 & u'_2 \\ u'_3 & 0 & u'_1 \\ u'_2 & u'_1 & 0 \end{pmatrix} = \begin{pmatrix} A & C & B \\ C & D & C \\ B & C & D \end{pmatrix}, \quad (11)$$

where  $A = hu_2$ ,  $B = h'u'_2$ ,  $C = h'u'_1 = h'u'_3$ , and  $D = hu_1 = hu_3$  have been assumed. This means that the residual symmetry in the singlet Higgs sector is  $Z_2$ , whereas that in the doublet Higgs sector is  $Z_3$ . This choice allows nonzero  $\theta_{13}$ , whereas the choice of Ref. [8] enforces  $\theta_{13} = 0$ .

The seesaw neutrino mass matrix is now

$$M_\nu = -M_D M_{\nu^c}^{-1} M_D^T = \frac{-f_D^2 v^2}{3 \det(M_{\nu^c})} \begin{pmatrix} AD - B^2 & C(B - A) & C(B - D) \\ C(B - A) & AD - C^2 & C^2 - BD \\ C(B - D) & C^2 - BD & D^2 - C^2 \end{pmatrix}, \quad (12)$$

where  $\det(M_{\nu^c}) = A(D^2 - C^2) + 2BC^2 - D(B^2 + C^2)$ . Redefining the parameters  $A, B, C, D$  to absorb the overall constant, we obtain the following neutrino mass matrix in the tribimaximal basis:

$$\mathcal{M}_\nu^{(1,2,3)} = \begin{pmatrix} D(A + D - 2B)/2 & C(2B - A - D)/\sqrt{2} & D(A - D)/2 \\ C(2B - A - D)/\sqrt{2} & AD - B^2 & C(D - A)/\sqrt{2} \\ D(A - D)/2 & C(D - A)/\sqrt{2} & (AD + D^2 + 2BD - 4C^2)/2 \end{pmatrix}. \quad (13)$$

This is obtained by first rotating with the  $3 \times 3$  unitary matrix of Eq. (9), which converts it to the  $(e, \mu, \tau)$  basis, then by Eq. (14) below. Note that for  $D = A$  and  $C = 0$ , this matrix becomes diagonal:  $m_1 = A(A - B)$ ,  $m_2 = A^2 - B^2$ ,  $m_3 = A(A + B)$ , which is the tribimaximal limit. Normal hierarchy of neutrino masses is obtained if  $B \simeq A$  and inverted hierarchy is obtained if  $B \simeq -2A$ .

The neutrino mixing matrix  $U$  has 4 parameters:  $s_{12}, s_{23}, s_{13}$  and  $\delta_{CP}$  [10]. We choose the convention  $U_{\tau 1}, U_{\tau 2}, U_{e 3}, U_{\mu 3} \rightarrow -U_{\tau 1}, -U_{\tau 2}, -U_{e 3}, -U_{\mu 3}$  to conform with that of the tribimaximal mixing matrix

$$U_{TB} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}. \quad (14)$$

then

$$\mathcal{M}_\nu^{(1,2,3)} = \begin{pmatrix} m_1 & m_6 & m_4 \\ m_6 & m_2 & m_5 \\ m_4 & m_5 & m_3 \end{pmatrix} = U_{TB}^T U \begin{pmatrix} e^{i\alpha_1} m'_1 & 0 & 0 \\ 0 & e^{i\alpha_2} m'_2 & 0 \\ 0 & 0 & m'_3 \end{pmatrix} U^T U_{TB}, \quad (15)$$

where  $m'_{1,2,3}$  are the physical neutrino masses, with

$$m'_2 = \sqrt{m_1'^2 + \Delta m_{21}^2}, \quad (16)$$

$$m'_3 = \sqrt{m_1'^2 + \Delta m_{21}^2/2 + \Delta m_{32}^2} \quad (\text{normal hierarchy}), \quad (17)$$

$$m'_3 = \sqrt{m_1'^2 + \Delta m_{21}^2/2 - \Delta m_{32}^2} \quad (\text{inverted hierarchy}). \quad (18)$$

If  $U$  and  $\alpha_{1,2}$  are known, then all  $m_{1,2,3,4,5,6}$  are functions only of  $m'_1$ .

In Ref. [9], the parameters  $A, B, C, D$  are assumed to be real, hence  $\delta_{CP}$  and  $\alpha_{1,2}$  are zero. We now consider  $C = E + iF$  to be complex. Thus  $m_{1,2,4}$  are real and  $m_{3,5,6}$  are complex. Since  $\mathcal{M}_\nu^{(1,2,3)}$  is in the tribimaximal basis, it can be diagonalized by an approximately diagonal unitary matrix. To first order, let

$$U_\epsilon = \begin{pmatrix} 1 & \epsilon_{12} & \epsilon_{13} \\ -\epsilon_{12}^* & 1 & \epsilon_{23} \\ -\epsilon_{13}^* & -\epsilon_{23}^* & 1 \end{pmatrix}, \quad (19)$$

then using

$$U_\epsilon \mathcal{M}_\nu^{(1,2,3)} U_\epsilon^T = \begin{pmatrix} e^{i\alpha'_1} m'_1 & 0 & 0 \\ 0 & e^{i\alpha'_2} m'_2 & 0 \\ 0 & 0 & e^{i\alpha'_3} m'_3 \end{pmatrix}, \quad (20)$$

we obtain  $\epsilon_{12}, \epsilon_{13}, \epsilon_{23}$  and  $\alpha'_{1,2,3}$  in terms of  $A, B, D, E, F$ . Using the four measured values  $\Delta m_{21}^2, \Delta m_{32}^2, s_{12}, s_{13}$ , and varying  $\delta_{CP}$ , we then obtain  $s_{23}$ , the physical relative Majorana phases  $\alpha_{1,2}$  in Eq. (15), and the effective Majorana neutrino mass in neutrinoless double beta decay, i.e.

$$m_{ee} = |U_{e1}^2 e^{i\alpha_1} m'_1 + U_{e2}^2 e^{i\alpha_2} m'_2 + U_{e3}^2 m'_3|. \quad (21)$$

Because of the structure of Eq. (13) from the  $T_7$  symmetry, even though the phase of the complex parameter  $C$  may be large, i.e.  $F/E$  large,  $\tan \delta_{CP}$  cannot be too large, because in the limit  $C = 0$ , there can be no  $CP$  violation, so any  $CP$  violating effect has to be proportional to  $F/D$  where  $D$  sets the neutrino mass scale. This is typically less than one because  $C \neq 0$  measures the deviation of  $\tan^2 \theta_{12}$  from the tribimaximal limit of  $1/2$ .

The unitary matrix  $U' = U_{TB}U_\epsilon^T$  has entries

$$U'_{e1} = \sqrt{\frac{2}{3}} + \sqrt{\frac{1}{3}}\epsilon_{12}, \quad U'_{e2} = \sqrt{\frac{1}{3}} - \sqrt{\frac{2}{3}}\epsilon_{12}^*, \quad U'_{e3} = -\sqrt{\frac{2}{3}}\epsilon_{13}^* - \sqrt{\frac{1}{3}}\epsilon_{23}^*, \quad (22)$$

$$U'_{\mu 3} = -\frac{1}{\sqrt{2}} + \frac{\epsilon_{13}^*}{\sqrt{6}} - \frac{\epsilon_{23}^*}{\sqrt{3}}, \quad U'_{\tau 3} = \frac{1}{\sqrt{2}} + \frac{\epsilon_{13}^*}{\sqrt{6}} - \frac{\epsilon_{23}^*}{\sqrt{3}}. \quad (23)$$

To obtain  $U$ , we rotate the phases of the  $\mu$  and  $\tau$  rows so that  $U'_{\mu 3}e^{-i\alpha'_3/2}$  is real and negative, and  $U'_{\tau 3}e^{-i\alpha_3/2}$  is real and positive. These phases are absorbed by the  $\mu$  and  $\tau$  leptons and are unobservable. We then rotate the  $\nu_{1,2}$  columns so that  $U'_{e1}e^{-i\alpha_3/2} = U_{e1}e^{i\alpha'_1/2}$  and  $U'_{e2}e^{-i\alpha_3/2} = U_{e2}e^{i\alpha'_2/2}$ , where  $U_{e1}$  and  $U_{e2}$  are real and positive. The physical relative Majorana phases of  $\nu_{1,2}$  are then  $\alpha_{1,2} = \alpha'_{1,2} + \alpha''_{1,2}$ . We now extract the three angles as well as  $\delta_{CP}$  as follows.

$$\tan^2 \theta_{12} = \left| \frac{U'_{e1}}{U'_{e2}} \right|^2 = \left( \frac{1}{2} \right) \frac{(1 - \sqrt{2} \operatorname{Re}(\epsilon_{12}))^2 + 2(\operatorname{Im}(\epsilon_{12}))^2}{(1 + \operatorname{Re}(\epsilon_{12})/\sqrt{2})^2 + (\operatorname{Im}(\epsilon_{12}))^2/2}, \quad (24)$$

$$\tan^2 \theta_{23} = \left| \frac{U'_{\mu 3}}{U'_{\tau 3}} \right|^2 = \frac{(1 - (\operatorname{Re}(\epsilon_{13} - \sqrt{2}\epsilon_{23})/\sqrt{3}))^2 + (\operatorname{Im}(\epsilon_{13} - \sqrt{2}\epsilon_{23}))^2/3}{(1 + (\operatorname{Re}(\epsilon_{13} - \sqrt{2}\epsilon_{23})/\sqrt{3})^2 + (\operatorname{Im}(\epsilon_{13} - \sqrt{2}\epsilon_{23}))^2/3)}, \quad (25)$$

$$\sin \theta_{13} e^{-i\delta_{CP}} = U'_{e3} e^{-i\alpha'_3/2}. \quad (26)$$

To see the approximate dependence of  $U_\epsilon$  on the  $T_7$  parameters  $A, B, D, E, F$ , we assume normal hierarchy and let

$$A = D + \delta_1, \quad B = D + \delta_2, \quad C = E + iF. \quad (27)$$

Expanding in  $\delta_{1,2}, E, F$  over  $D$ , we then have

$$m_1 = \frac{D}{2}(\delta_1 - 2\delta_2), \quad m_2 = D(\delta_1 - 2\delta_2) - \delta_2^2, \quad m_3 = 2D^2 + \frac{D}{2}(\delta_1 + 2\delta_2), \quad (28)$$

$$m'_1 = m_1 - \frac{\delta_1^2}{8}, \quad m'_2 = m_2, \quad m'_3 = m_3, \quad (29)$$

$$m_4 = \frac{D}{2}\delta_1, \quad m_5 = -\frac{\delta_1}{\sqrt{2}}(E + iF), \quad m_6 = -\frac{E + iF}{\sqrt{2}}(\delta_1 - 2\delta_2), \quad (30)$$

$$\operatorname{Re}(\epsilon_{12}) = \frac{\sqrt{2}E}{D} \left( 1 - \frac{\delta_1^2}{4D(\delta_1 - 2\delta_2)} \right) \left( 1 + \frac{\delta_1^2 - 8\delta_2^2}{4D(\delta_1 - 2\delta_2)} \right)^{-1}, \quad (31)$$

$$\operatorname{Im}(\epsilon_{12}) = \frac{\sqrt{2}F}{3D} \left( 1 - \frac{\delta_1^2}{4D(\delta_1 - 2\delta_2)} \right) \left( 1 - \frac{\delta_1^2 + 8\delta_2^2}{12D(\delta_1 - 2\delta_2)} \right)^{-1}, \quad (32)$$

$$\epsilon_{13} = -\frac{\delta_1}{4D} \left( 1 + \frac{\delta_2}{D} \right)^{-1}, \quad \epsilon_{23} = \frac{\delta_1}{2\sqrt{2}D^2}(E + iF). \quad (33)$$

Using Eq. (26), and neglecting  $\alpha'_3$ , we obtain

$$\tan \delta_{CP} = \frac{F}{D} \left(1 + \frac{\delta_2}{D}\right) \left[1 - \frac{E}{D} \left(1 + \frac{\delta_2}{D}\right)\right]^{-1}. \quad (34)$$

Assuming inverted hierarchy, we let

$$A = D + \delta_1, \quad B = -2D + \delta_2, \quad C = E + iF, \quad (35)$$

then

$$m'_1 = m_1 = 3D^2 + \frac{D}{2}(\delta_1 - 2\delta_2), \quad m'_2 = m_2 = -3D^2 + D(\delta_1 + 4\delta_2), \quad (36)$$

$$m'_3 = m_3 = -D^2 + \frac{D}{2}(\delta_1 + 2\delta_2), \quad (37)$$

$$m_4 = \frac{D}{2}\delta_1, \quad m_5 = -\frac{\delta_1}{\sqrt{2}}(E + iF), \quad m_6 = -\frac{E + iF}{\sqrt{2}}(6D + \delta_1 - 2\delta_2), \quad (38)$$

$$Re(\epsilon_{12}) = -\frac{E}{D\sqrt{2}} \left(1 + \frac{\delta_1}{6D} - \frac{\delta_2}{3D}\right) \left(1 - \frac{\delta_1}{12D} - \frac{5\delta_2}{6D}\right)^{-1}, \quad (39)$$

$$Im(\epsilon_{12}) = \frac{2\sqrt{2}F}{\delta_1 + 2\delta_2} \left(1 + \frac{\delta_1}{6D} - \frac{\delta_2}{3D}\right), \quad (40)$$

$$Re(\epsilon_{13}) = \frac{\delta_1}{8D} \left(1 - \frac{\delta_2}{2D}\right)^{-1}, \quad Im(\epsilon_{13}) = -\frac{F\delta_1}{16\sqrt{2}D^2}, \quad (41)$$

and  $\epsilon_{23}$  is determined by

$$-\epsilon_{23}^* m_2 + \epsilon_{23} m_3 = -m_5 + \epsilon_{12}^* m_4 + \epsilon_{13}^* m_6 - \epsilon_{13}^* \epsilon_{12}^* m_1. \quad (42)$$

For our numerical analysis, we set

$$\Delta m_{21}^2 = 7.59 \times 10^{-5} \text{ eV}^2, \quad \Delta m_{32}^2 = 2.45 \times 10^{-3} \text{ eV}^2, \quad (43)$$

$$\sin^2 2\theta_{12} = 0.87, \quad \sin^2 2\theta_{13} = 0.092. \quad (44)$$

We then diagonalize Eq. (15) exactly and scan for solutions satisfying the above experimental inputs. Assuming normal hierarchy, we find  $\sin^2 2\theta_{23}$  to range from 0.9501 for  $\delta_{CP} = 0$  to 0.9505 for  $|\tan \delta_{CP}| = 0.2$ , as shown in Fig. 1. This is an imperceptible change, so our model prediction for  $\sin^2 2\theta_{23}$  is basically unchanged from the real case. We show the absolute



values  $|A|$ ,  $|B|$ ,  $|D|$ , and  $|C|$  as functions of  $\sin^2 2\theta_{23}$  in Fig. 2, and  $E$  versus  $F$  in Fig. 3. As expected,  $F/E$  may be large, but  $\tan \delta_{CP} \simeq F/D$  remains small. We then plot the three physical neutrino masses  $m'_{1,2,3}$  as well as  $m_{ee}$  as functions of  $|\tan \delta_{CP}|$  in Fig. 4, and the Majorana phases  $\alpha_{1,2}$  versus  $|\tan \delta_{CP}|$  in Fig. 5. For inverted hierarchy, we show in Figs. 6 to 10 the corresponding plots. We note again that  $\tan \delta_{CP}$  is small, but now  $\alpha_{1,2}$  are much larger. This can be seen from Eq. (40) versus Eq. (32).

In conclusion, we have studied how the  $T_7$  model of Ref. [9] allows  $CP$  violation in the neutrino mixing matrix. There are three real parameters  $A, B, D$  and one complex parameter  $C = E + iF$ , from which nine physical observables may be derived. Given the experimental inputs  $\Delta m_{21}^2$ ,  $\Delta m_{32}^2$ ,  $\sin^2 2\theta_{12}$ , and the recently measured  $\sin^2 2\theta_{13}$ , the remaining five observables depend on only one variable which we choose to be  $\delta_{CP}$ . Because of the structure of the neutrino mass matrix constrained by  $T_7$ , even if  $C$  has a large phase, i.e.  $F/E$  is large,  $\tan \delta_{CP}$  remains small. For  $\sin^2 2\theta_{13} = 0.092$  and  $\sin^2 2\theta_{12} = 0.87$ , we find  $\sin^2 2\theta_{23}$  to be essentially fixed at 0.95 as  $|\tan \delta_{CP}|$  changes from 0.0 to 0.2. The Majorana phases  $\alpha_{1,2}$  are comparable to  $\delta_{CP}$  in magnitude for normal hierarchy, but are much larger for inverted hierarchy.

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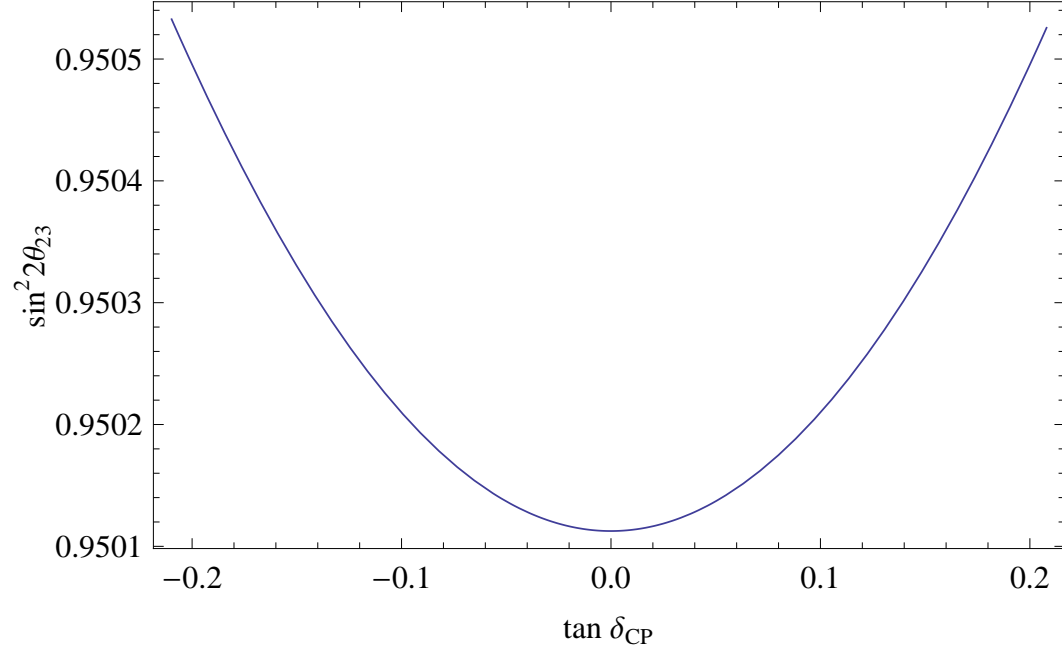


Figure 1:  $\sin^2 2\theta_{23}$  versus  $\tan \delta_{CP}$  for normal hierarchy.

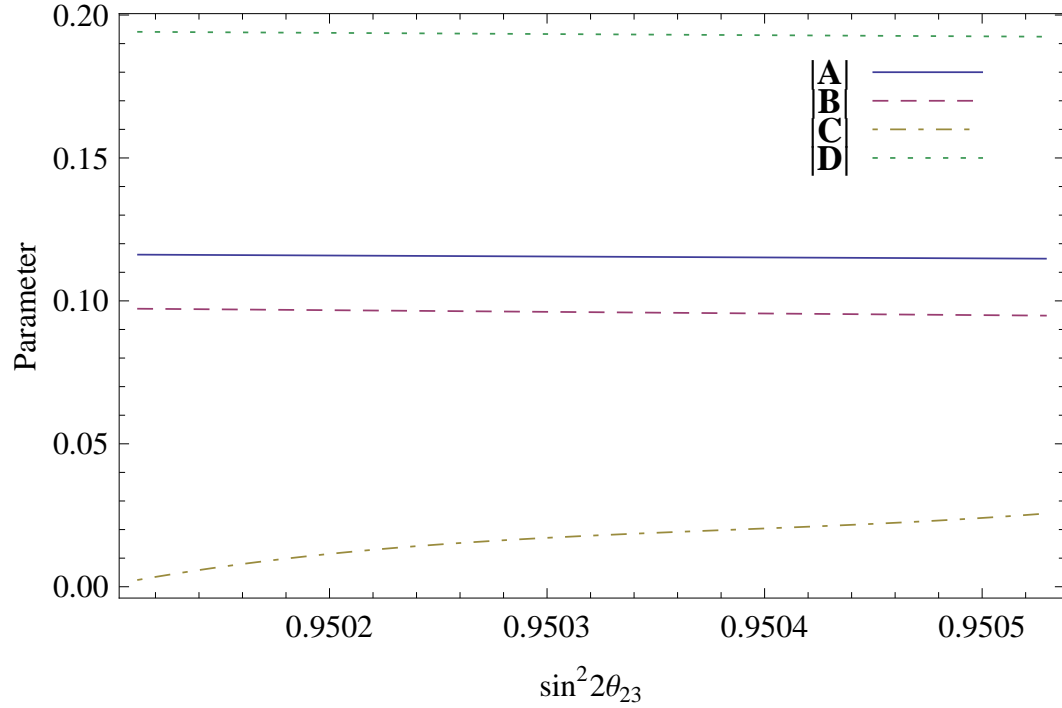


Figure 2:  $T_7$  parameters for normal hierarchy.

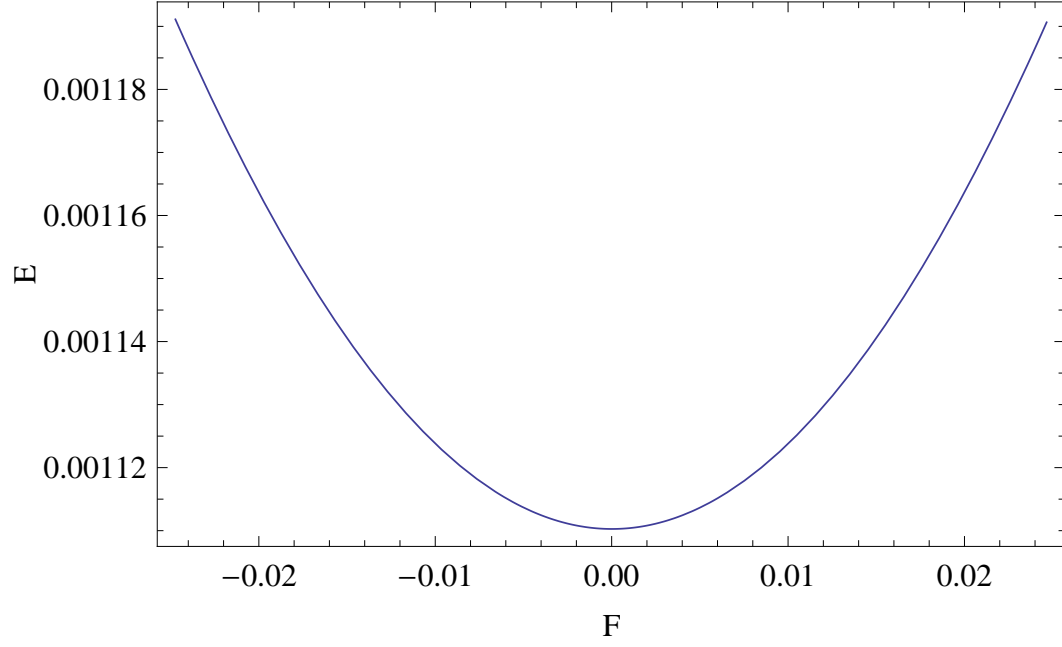


Figure 3:  $T_7$  complex parameters  $E$  versus  $F$  for normal hierarchy.

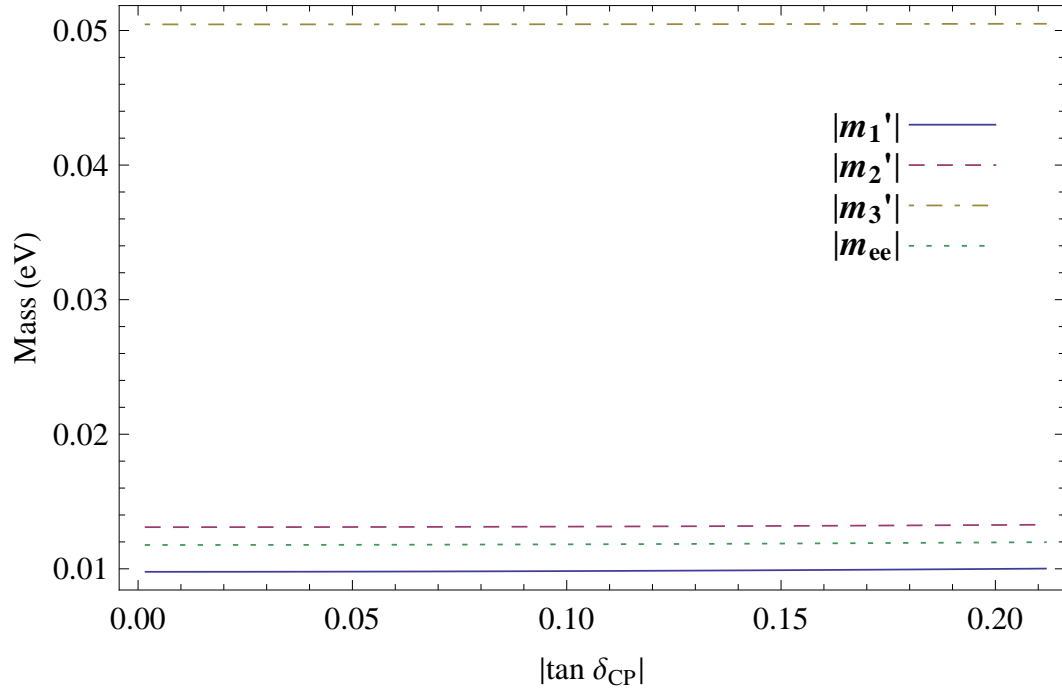


Figure 4: Physical neutrino masses and the effective neutrino mass  $m_{ee}$  in neutrinoless double beta decay for normal hierarchy.

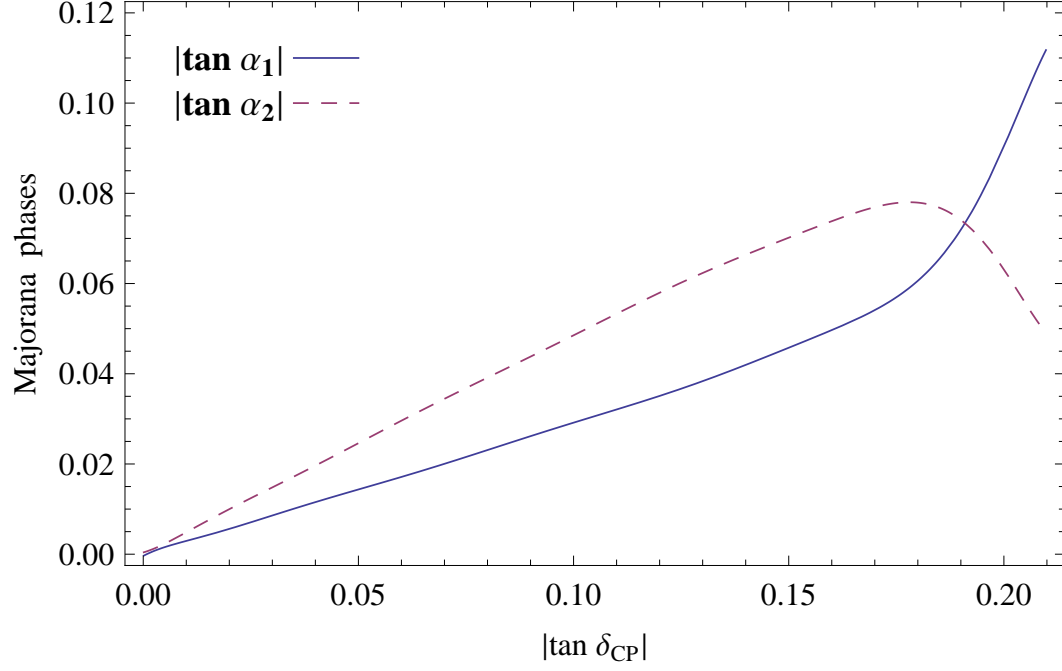


Figure 5: Majorana phases  $|\tan \alpha_1|$  and  $|\tan \alpha_2|$  versus  $|\tan \delta_{\text{CP}}|$  for normal hierarchy.

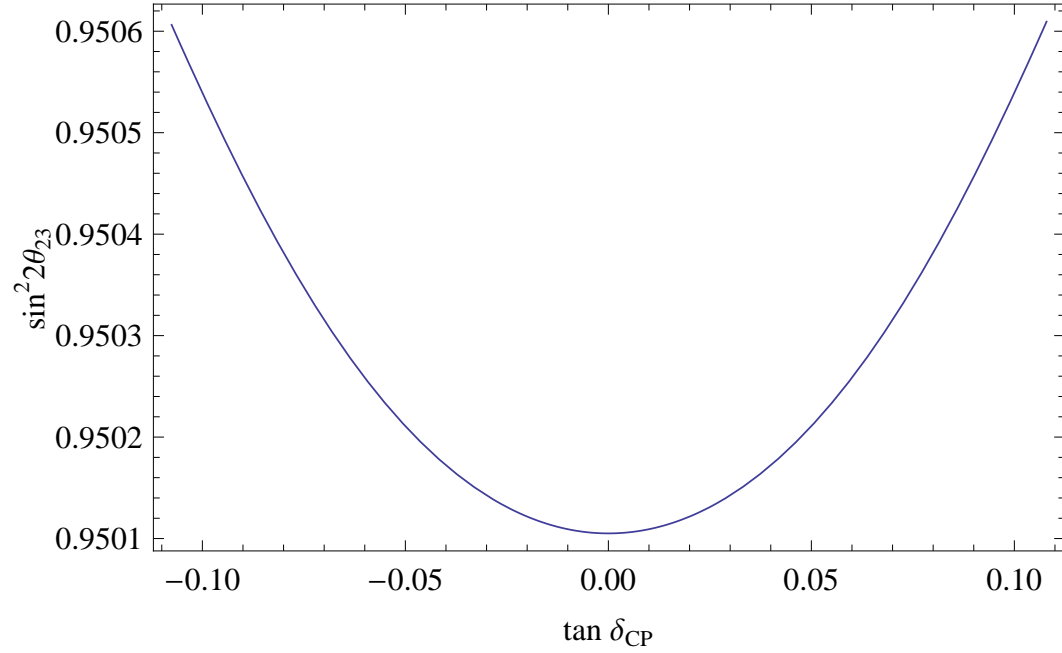


Figure 6:  $\sin^2 2\theta_{23}$  versus  $\tan \delta_{\text{CP}}$  for inverted hierarchy.

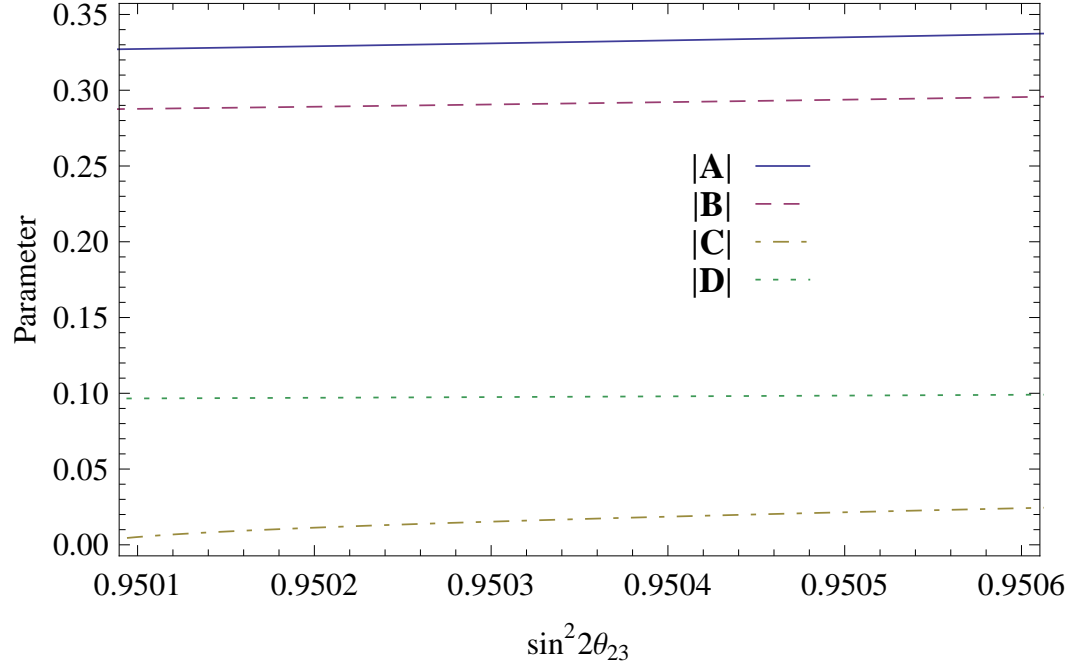


Figure 7:  $T_7$  parameters for inverted hierarchy.

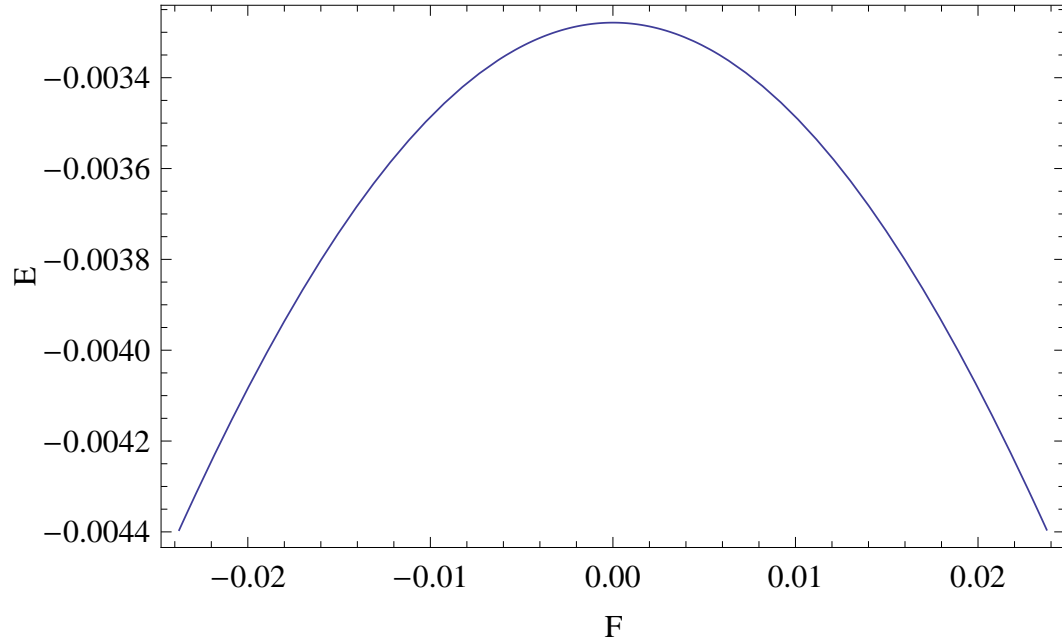


Figure 8:  $T_7$  complex parameters  $E$  versus  $F$  for inverted hierarchy.

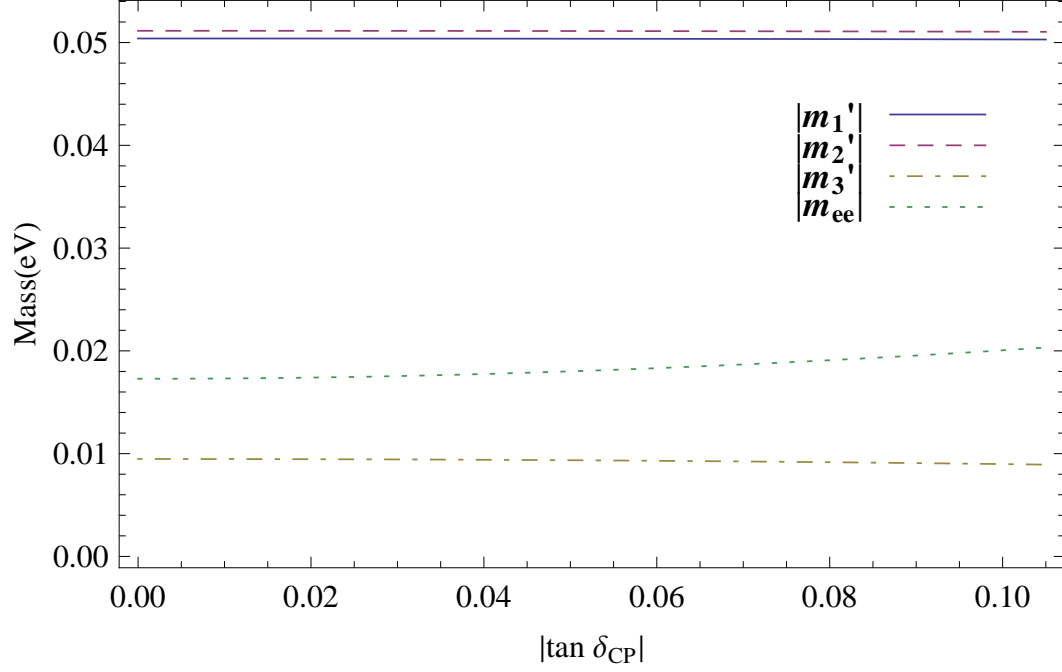


Figure 9: Physical neutrino masses and the effective neutrino mass  $m_{ee}$  in neutrinoless double beta decay for inverted hierarchy.

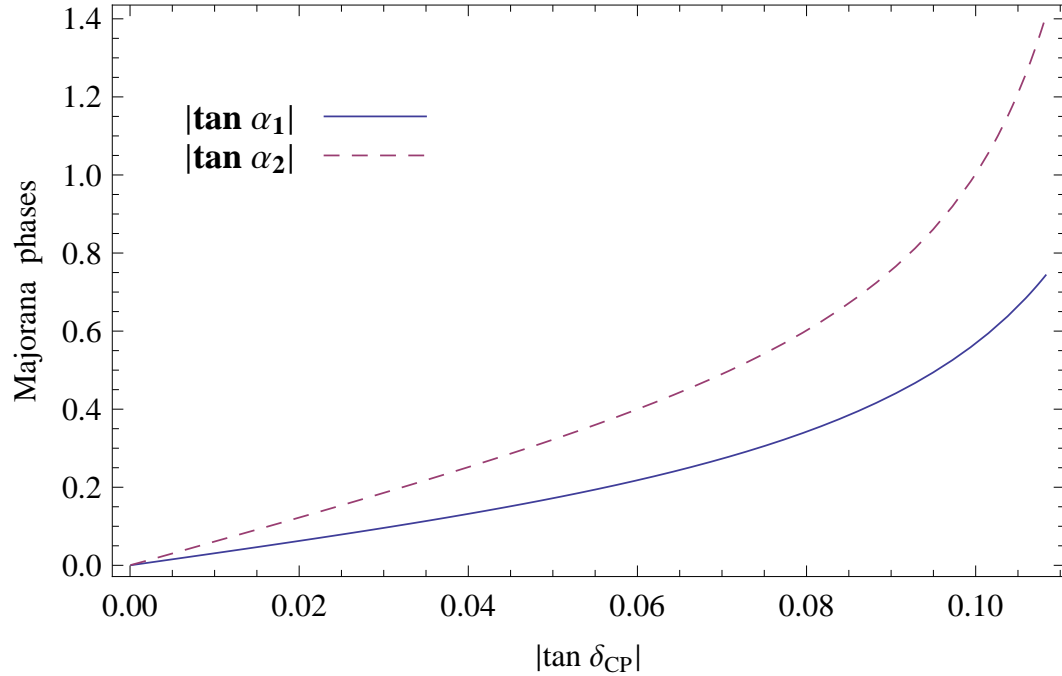


Figure 10: Majorana phases  $|\tan \alpha_1|$  and  $|\tan \alpha_2|$  versus  $|\tan \delta_{CP}|$  for inverted hierarchy.